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SOME CONTINUOUS FUNCTIONS RELATED TO CORNER POLYHEDRA, II

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The group problem on the unit interval is developed, with and without continuous variables. The connection with cutting planes, or valid inequalities, is reviewed. Certain desirable properties of valid inequalities, such as minimality and extremality are developed, and the connection between valid inequalities for $\mathrm{P}(I, u_0)$ and $\mathrm{P}^+_-(I, u_0)$ is developed. A class of functions is shown to give extreme valid inequalities for $\mathrm{P}^+_-(I, u_0)$ and for certain subsets U of I. A method is used to generate such functions. These functions give faces of certain corner polyhedra. Other functions which do not immediately give extreme valid inequalities are altered to construct a class of faces for certain corner polyhedra. This class of faces grows exponentially as the size of the group grows.

1. Review of the problems

This paper follows a previous paper [4] but will be self-contained except for proofs of some theorems from [4].

1.1. *The problems* $\mathrm{P}(U, u_0)$ *and* $\mathrm{P}^+_-(U, u_0)$

Let I be the group formed by the real numbers on the interval [0, 1] with addition modulo 1. Let U be a subset of I and let t be an integer-valued function on U such that (i) $t(u) \geq 0$ for all $u \in U$, and (ii) t has a *finite support*, that is $t(u) > 0$ only for a finite subset U_t of U.

We say that the function t is a *solution to the problem* $\mathrm{P}(U, u_0)$, for $u_0 \in I \setminus \{0\}$, if

$$\sum_{u \in U} u\, t(u) = u_0 \,. \tag{1.1}$$

Here, of course, addition and multiplication are taken modulo 1. Let $T(U, u_0)$ denote the set of all such solutions t to $\mathrm{P}(U, u_0)$.

Correspondingly, the *problem* $\mathrm{P}^+_-(U, u_0)$ *has solutions* $t' = (t, s^+, s^-)$ satisfying

$$\sum_{u \in U} u\, t(u) + \mathcal{F}(s^+) - \mathcal{F}(s^-) = u_0 \ , \tag{1.2}$$

where t is, as before, a non-negative integer valued function on U with a finite support, where s^+, s^- are non-negative real numbers, and where $\mathcal{F}(x)$ denotes the element of I given by taking the fractional part of a real number x. Let $T^+_-(U, u_0)$ denote the set of solutions $t' = (t, s^+, s^-)$ to $\mathrm{P}^+_-(U, u_0)$.

The notation $u \in I$ will mean that u is a member of the group I so that arithmetic is always modulo 1. If we want to consider u as a point on the real line with real arithmetic, we will write $|u|$. Thus, $|u|$ and $\mathcal{F}(x)$ are mappings in opposite directions between I and the reals and, in fact, $\mathcal{F}(|u|) = u$ but x and $|\mathcal{F}(x)|$ may differ by an integer.

1.2. *Inequalities*

1.2.1. *Valid inequalities*

For any problem $\mathrm{P}(U, u_0)$, we have so far defined the solution set $T(U, u_0)$. A *valid inequality for the problem* $\mathrm{P}(U, u_0)$ is a real-valued function π defined for all $u \in I$ such that

$$\pi(0) = 0, \quad \pi(u) \geq 0, \quad u \in I \, , \tag{1.3}$$

and

$$\sum_{u \in U} \pi(u)\, t(u) \geq 1 \, , \quad t \in T(U, u_0) \, . \tag{1.4}$$

For the problem $\mathrm{P}^+_-(U, u_0)$, $\pi' = (\pi, \pi^+, \pi^-)$ is a *valid inequality for* $\mathrm{P}^+_-(U, u_0)$ when π is a real-valued function on I satisfying (1.3), and π^+, π^- are non-negative real numbers such that

$$\sum_{u \in U} \pi(u)\, t(u) + \pi^+ s^+ + \pi^- s^- \geq 1 \, , \quad t' \in T^+_-(U, u_0) \, . \tag{1.5}$$

A valid inequality (π, π^+, π^-) for $\mathrm{P}^+_-(I, u_0)$ can be used to give a valid inequality for $\mathrm{P}(U, u_0)$ or $\mathrm{P}^+_-(U, u_0)$ for any subset U of I. For example,

$\Sigma\pi(u)\, t(u) \geq 1$ is clearly true for any $t \in T(U, u_0)$ since that t can be extended to a function t' belonging to $T(I, u_0)$ by letting $t'(u) = 0$ for $u \in I\backslash U$. Thus, the problem $P^+_-(I, u_0)$ acts as a master problem for all cyclic group problems in the same way that the master problem in [3] was a group problem with all group elements present. This fact is the main reason for studying the case $U = I$ in such detail in Section 2. However, the next two properties of valid inequalities do not necessarily carry over to subsets U of I.

1.2.2. *Minimal valid inequalities*

A valid inequality π for $P(U, u_0)$ is a *minimal valid inequality* for $P(U, u_0)$ if there is no other valid inequality ρ for $P(U, u_0)$ satisfying $\rho(u) < \pi(U)$, where $\rho(U) < \pi(U)$ is defined to mean $\rho(u) \leq \pi(u)$ for all $u \in U$ and $\rho(u) < \pi(u)$ for at least one $u \in U$. A valid inequality π' for $P^+_-(U, u_0)$ is a minimal valid inequality for $P^+_-(U, u_0)$ if there is no other valid inequality ρ' for $P^+_-(U, u_0)$ satisfying $\rho'(U) < \pi'(U)$, where $\rho'(U) < \pi(U)$ is defined to mean

$$\rho^+ \leq \pi^+ \,, \quad \rho^- \leq \pi^- \,, \quad \rho(u) \leq \pi(u) \,, \quad u \in U \,,$$

with strict inequality holding for at least one of the above inequalities.

The minimal valid inequalities are important because a valid inequality which is not minimal is implied by some other valid inequality. Note that we have scaled the inequalities to have a right-hand side equal to one, and minimality is always with respect to that scaling.

1.2.3. *Extreme valid inequalities*

A valid inequality π for $P(U, u_0)$ is an *extreme valid inequality* for $P(U, u_0)$ if π can not be written as $\pi = \frac{1}{2}\rho + \frac{1}{2}\sigma$ for $\rho \neq \sigma$, where ρ, σ are valid inequalities for $P(U, u_0)$.

A valid inequality $\pi' = (\pi, \pi^+, \pi^-)$ for $P^+_-(U, u_0)$ is an *extreme valid inequality* for $P^+_-(U, u_0)$ if π' cannot be written as $\pi' = \frac{1}{2}\rho + \frac{1}{2}\sigma'$ for $\rho' \neq \sigma'$, where ρ', σ' are valid inequalities for $P^+_-(U, u_0)$.

Theorem 1.1 [4, Theorem 1.1]. *The extreme valid inequalities are minimal valid inequalities.*

These inequalities are in some sense "the best" possible since they cannot be derived from any other valid inequalities.

1.2.4. *Subadditive valid inequalities*

A valid inequality π for $P(U, u_0)$ is a subadditive valid inequality for $P(U, u_0)$ if

$$\pi(u) + \pi(v) \geq \pi(u + v) \qquad \text{whenever } u, v, u + v \in U\,. \tag{1.6}$$

For a valid inequality π' for $P^{+}_{-}(U, u_0)$ to be subadditive, we require, in addition to (6),

$$\pi(u) + \pi^{+}\,|v - u| \geq \pi(v) \quad \text{whenever } u, v \in U \text{ and } |u| < |v|\,, \tag{1.7}$$

$$\pi(u) + \pi^{-}\,|u - v| \geq \pi(v) \quad \text{whenever } u, v \in U \text{ and } |u| > |v|\,. \tag{1.8}$$

Theorem 1.2 [4, Theorem 1.2]. *The minimal valid inequalities are subadditive valid inequalities.*

Thus Theorems 1.1 and 1.2 prove the following sequence of inclusions: The set of valid inequalities include the subadditive valid inequalities which include minimal valid inequalities which include extreme valid inequalities. The subadditive valid inequalities form a convex set contained in the larger convex set of valid inequalities.

Theorem 1.3 [4, Theorem 1.3]. *If* π *(or* π'*) is extreme among the subadditive valid inequalities for* $P(U, u_0)$ *(or* $P^{+}_{-}(U, u_0)$*), that is,* π *(or* π'*) is not the midpoint of any two different subadditive valid inequalities, and if* π *(or* π'*) is also a minimal valid inequality, then it is an extreme valid inequality.*

Thus Theorem 1.3 says that the extreme points of the set of subadditive valid inequalities include all the extreme valid inequalities. Further. among the extreme subadditive valid inequalities those which are extreme valid inequalities are the minimal ones. This fact allows us to actually find the extreme valid inequalities for some problems.

1.3. *Subadditivity for subgroups U*

The problems for which we can find extreme valid inequalities are $P(U, u_0)$ or $P^{+}_{-}(U, u_0)$ where U is a nonempty subgroup of I. We permit $U = I$ and note that 0 is always in U. We will say that a function π defined on I is *subadditive on a subgroup U* of I if

$$\pi(0) = 0\,, \quad \pi(u) \geq 0\,, \quad u \in I\,,$$
$$\pi(u) + \pi(v) \geq \pi(u+v)\,, \quad u, v \in U\,.$$

The function π is not assumed to be a valid inequality.

Theorem 1.4 [4, Theorem 1.5]. *If π is a subadditive function on a subgroup U of I and if $\pi(u_0) \geq 1$ for some $u_0 \in U$, $u_0 \neq 0$, then π is a valid inequality for* $\mathrm{P}(U, u_0)$. *In fact, the subadditive valid inequalities for* $\mathrm{P}(U, u_0)$ *are precisely the subadditive functions π satisfying $\pi(u_0) \geq 1$. Furthermore, if π is a subadditive function on U and $\pi(u_0) > 0$ for some $u_0 \in U$, then π^* defined by*

$$\pi^*(u) = \pi(u)/\pi(u_0)\,, \quad u \in I\,, \tag{1.9}$$

is a valid inequality for $\mathrm{P}(U, u_0)$.

Thus Theorem 1.4 establishes the close connection between subadditive functions on U and valid inequalities.

The analogous theorem for $\mathrm{P}_-^+(U, u_0)$ will now be developed. Define $\pi' = (\pi, \pi^+, \pi^-)$ to be an *extended subadditive function on a subgroup U of I* if π is subadditive on U and if, in addition,

$$\pi^+ |u| \geq \pi(u)\,, \qquad u \in U\,, \tag{1.10}$$

$$\pi^- |u| \geq \pi(-u)\,, \quad -u \in U\,. \tag{1.11}$$

Theorem 1.5 [4, Theorem 1.5′]. *If π' is an extended subadditive function on a subgroup U of I, if $u_0 \in I$, $u_0 \neq 0$, and if both of the following hold:*

$$\pi(u) + \pi^+ |u_0 - u| \geq 1 \quad \textit{whenever } u \in U \textit{ and } |u| \leq |u_0|\,, \tag{1.12}$$

$$\pi(u) + \pi^- |u - u_0| \geq 1 \quad \textit{whenever } u \in U \textit{ and } |u| \leq |u_0|\,. \tag{1.13}$$

then π' is a valid inequality for $\mathrm{P}_-^+(U, u_0)$. *In fact, the subadditive valid inequalities are precisely the extended subadditive functions which satisfy* (1.12) *and* (1.13).

1.4. *Minimality for subgroups U*

Theorem 1.6 [4, Theorem 1.6]. *If U is a subgroup of I with $u_0 \in U$ and if π is a valid inequality for* $\mathrm{P}(U, u_0)$, *then π is a minimal valid inequality if and only if*

$$\pi(u) + \pi(u_0 - u) = 1\,, \quad u \in U\,. \tag{1.14}$$

This condition imposes a peculiar symmetry on π so that $\pi(u)$ for $\frac{1}{2}u_0 < u < u_0$ is determined by $\pi(u)$ on $[0, \frac{1}{2}u_0]$, for example.

1.5. *The problem* $\mathrm{P}(G_n, u_0)$, $u_0 \in G_n$

Let G_n denote the subset

$$G_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\right\}$$

of I. The elements of G_n will be denoted $g_i = \mathcal{F}(i/n)$. Each set G_n for $n \geq 1$ is a subgroup of I. By virtue of G_n being a subgroup, the results of Sections 1.3 and 1.4 apply to the present section.

The results from Sections 1.3 and 1.4 are specialized in the following theorem:

Theorem 1.7 [4, Theorem 2.2]. *The extreme valid inequalities for* $\mathrm{P}(G_n, u_0)$, $u_0 \in G_n$, *are the extreme points of the solutions to*

$$\pi(0) = 0\,, \qquad \pi(g_i) \geq 0\,, \tag{1.15}$$

$$\pi(g_i) + \pi(g_j) \geq \pi(g_i + g_j)\,, \tag{1.16}$$

$$\pi(u_0) \geq 1\,, \tag{1.17}$$

which satisfy the additional equations

$$\pi(g_i) + \pi(u_0 - g_i) = 1\,, \quad g_i \in G_n\,. \tag{1.18}$$

In particular, (1.18) *implies* $\pi(u_0) = 1$ *since* $\pi(0) = 0$.

1.6. *The problem* $\mathrm{P}_{-}^{+}(G_n, u_0)$, $u_0 \in I$

The condition (1.2) now becomes

$$g_1 t(g_1) + \ldots + g_{n-1}\ t(g_{n-1}) + \mathcal{F}(s^+) - \mathcal{F}(s^-) = u_0 \ ,$$

where $g_i = \mathcal{F}(i/n)$ as before and where the $t(g_i)$ must be nonnegative integers and s^+, s^- must be nonnegative real values. We no longer confine u_0 to be in G_n. Let $L(u_0)$ and $R(u_0)$ denote the points of G_n immediately to the left and to the right of u_0, respectively. If u_0 happens to be in G_n, then $L(u_0) = R(u_0) = u_0$.

Theorem 1.8 [4, Theorem 2.2′]. *The extreme valid inequalities* π' *for* $\mathrm{P}^+_-(G_n, u_0)$, $u_0 \in I$, *are the extreme points of the solutions to the system of linear equations and inequalities* (1.15) *and* (1.16) *and all of the following:*

$$\pi^+(1/n) \geq \pi(g_1)\ , \quad g_1 = \mathcal{F}(1/n)\ , \tag{1.18}$$

$$\pi^-(1/n) \geq \pi(g_{n-1})\ , \quad g_{n-1} = \mathcal{F}((n-1)/n)\ . \tag{1.19}$$

$$\pi(L(u_0)) + \pi^+ |u_0 - L(u_0)| = 1\ , \tag{1.20}$$

$$\pi(R(u_0)) + \pi^- |R(u_0) - u_0| = 1\ , \tag{1.21}$$

$$\pi(g_i) + \pi(L(u_0) - g_i) = \pi(L(u_0))\ , \quad g_i \in G_n\ ,$$

or (1.22)

$$\pi(g_i) + \pi(R(u_0) - g_i) = \pi(R(u_0))\ , \quad g_i \in G_n\ .$$

1.7. *Valid inequalities for* $\mathrm{P}(U, u_0)$

We now connect the results about $\mathrm{P}(G_n, u_0)$ with the general problem $\mathrm{P}(U, u_0)$. Here, U can be any subset of the unit interval, including the interval I itself.

Theorem 1.9 [4, Theorem 3.1]. *Let* π *be a subadditive function on* G_n. *Define*

$$\pi(u) = n\,(|u - L(u)|\ \ \pi(R(u)) + |R(u) - u|\ \ \pi(L(u)) \ , \qquad u \in I \setminus G_n\ . \tag{1.23}$$

Then π *is a subadditive function on* I, *and* π^* *defined on* I *by*

$$\pi^*(u) = \pi(u)/\pi(u_0)\ , \quad u \in I\ ,$$

is a valid inequality for any $\mathrm{P}(U, u_0)$, U *a subset of* I, *provided* $\pi(u_0) > 0$.

Thus Theorem 1.9 says that valid inequalities can be obtained simply by connecting the points $(g_n, \pi(g_n))$ by straight line segments.

1.8. *Valid inequalities for* $P^+_-(U, u_0)$

From valid inequalities for $P^+_-(G_n, u_0)$, a different method for generating valid inequalities for $P^+_-(U, u_0)$ is available. This method will be referred to as the *two-slope fill-in*:

Theorem 1.10 [4, Theorem 3.3]. *Let* $\pi' = (\pi, \pi^+, \pi^-)$ *be an extended subadditive function on* G_n. *Define* $\pi(u)$ *for* $u \in I\backslash G_n$ *by*

$$\pi(u) = \min\{\pi(L(u)) + \pi^+ |u - L(u)|, \quad \pi(R(u)) + \pi^- |R(u) - u|\} . \tag{1.24}$$

Then π' *is an extended subadditive function on I, and* ρ' *defined by*

$$\rho' = (\pi, \pi^+, \pi^-)/\pi(u_0)$$

is a valid inequality for $P^+_-(U, u_0)$ *provided* $\pi(u_0) > 0$.

Theorem 1.8 shows how to compute faces for $P^+_-(G_n, u_0)$ and Theorem 1.10 shows how to use them to generate valid inequalities for any U. Table 2 of [4] was obtained using Theorem 1.8, and we will frequently refer to the two-slope fill-in of those faces.

2. The problems $P(I, u_0)$ and $P^+_-(I, u_0)$

2.1. *Problem definitions*

Let the set U now be the entire interval I. The problem $P(I, u_0)$ involves the congruence

$$\sum_{u \in I} u\, t(u) = u_0 , \tag{2.1}$$

and $P^+_-(I, u_0)$ has the constraint

$$\sum_{u \in I} u\, t(u) + \mathcal{F}(s^+) - \mathcal{F}(s^-) = u_0 , \tag{2.2}$$

where t is a non-negative integer valued function on I having finite support.

The present section intends to reveal something about the extreme valid inequalities for those problems. Such information could be useful in dealing with problems involving subsets of I. The relation to $P(U, u_0)$ is the same as the relation between the master polyhedra and the corner polyhedra corresponding to subsets of a group [3]. Here, every finite cyclic group G_n is a subset of I. In particular, if π is a valid inequality for $P(I, u_0)$, then trivially π is also a valid inequality for $P(U, u_0)$ for every subset U of I, including all cyclic groups $U = G_n$ or subset U of G_n. Furthermore, if π' is a valid inequality for $P^+_-(I, u_0)$, then π is a valid inequality for $P(U, u_0)$, (π, π^+) is a valid inequality for $P^+(U, u_0)$, (π, π^-) is a valid inequality for $P_-(U, u_0)$, and $\pi' = (\pi, \pi^+, \pi^-)$ is a valid inequality for $P^+_-(U, u_0)$ for any subset U of I.

The property of being a valid inequality is hereditary, that is, if π is a valid inequality for $P(S, u_0)$, then it is also valid for any $P(S', u_0)$ with $S' \subset S$. Subadditivity for a valid inequality is also hereditary. However, minimality and extremeness are *not* hereditary properties. That is, π can be a minimal or extreme valid inequality for $P(U, u_0)$ and still not be for $P(U', u_0)$ with $U' \subset U$.

2.2. *Properties and relations between* $P(I, u_0)$ *and* $P^+_-(I, u_0)$

Property 2.1. *If* $\pi' = (\pi, \pi^+, \pi^-)$ *is a valid inequality for* $P^+_-(I, u_0)$, *then* π *is a valid inequality for* $P(I, u_0)$.

Proof. If π is not a valid inequality for $P(I, u_0)$, then there is a t satisfying (1) with $\Sigma\pi(u)\, t(u) < 1$. Clearly $(t, 0, 0)$ solves (2.2) as well, contradicting π' being a valid inequality for $P^+_-(I, u_0)$, and completing the proof.

Recall that we define $|u|$ as the real number corresponding to $u \in I$. We can then define right and left limits,

$$\lim_{u \downarrow u_0}, \quad \lim_{u \uparrow u_0},$$

as the point $|u|$ approaches $|u_0|$ on the real line from the right ($|u| > |u_0|$) or from the left ($|u| < |u_0|$), respectively.

Property 2.2. *If π is a valid inequality for* $\mathrm{P}(I, u_0)$ *and if*

$$l^+ = \lim_{u \downarrow 0} \pi(u)/|u| \ , \qquad l^- = \lim_{u \uparrow 1} \{\pi(u)/(1 - |u|)\}$$

both exist (that is, if π has right and left "derivatives" at 0 *and* 1, *respectively), then $\pi' = (\pi, l^+, l^-)$ is a valid inequality for* $\mathrm{P}_-^+(I, u_0)$.

Proof. Suppose $t' = (t, s^+, s^-)$ solves (2.2) but

$$\sum_{u \in I} \pi(u)\, t(u) + l^+ s^+ + l^- s^- = 1 - \epsilon \ , \quad \epsilon > 0 \ .$$

We can assume that only one of s^+, s^- is positive, say $s^+ > 0$ and $s^- = 0$, since otherwise both s^+ and s^- could be reduced until one reaches zero. Choose an integer M large enough that

$$\left| l^+ - \frac{\pi(\mathcal{F}(s^+/M))}{s^+/M} \right| < \frac{\epsilon}{s^+} \ ,$$

which can be done since l^+ exists. Let

$$t_1(u) = \begin{cases} t(u) & u \neq s^+/M \ , \\ t(u) + M \ , & u = s^+/M \ . \end{cases}$$

Clearly t_1 satisfies (2.1) since t' satisfied (2.2). But

$$\begin{aligned} \sum_{u \in I} \pi(u)\, t_1(u) &= \sum_{u \in I} \pi(u)\, t(u) + M\, \pi(s^+/M) \\ &< \sum_{u \in I} \pi(u)\, t(u) + l^+ s^+ + \epsilon = 1 \ , \end{aligned}$$

contradicting π being a valid inequality for $\mathrm{P}(I, u_0)$.

Lemma 2.3. *If π is a subadditive function on I and if*

$$\limsup_{u \downarrow 0} \{\pi(u)/|u|\} = \beta < \infty \ ,$$

then

$$\lim_{u \downarrow 0} \{\pi(u)/|u|\} = \beta \ .$$

Proof. If the limit does not exist, then

$$\liminf_{u \downarrow 0} \{\pi(u)/|u|\} \neq \beta ,$$

that is, there are points v arbitrarily close to 0 with $\pi(v)/|v| \leq \alpha < \beta$. By the limsup being β, there are also points u arbitrarily close to 0 with $\pi(u)/|u| > \alpha$. Choose any such u and choose $0 < v < u$ with $\pi(v)/|v| \leq \alpha < \beta$. Then $|u|$ can be written as an integer multiple of $|v|$ and a remainder

$$|u| = \lfloor u/v \rfloor \, |v| + \gamma(u) , \quad 0 \leq \gamma(u) < |v| .$$

Since π is subadditive on I,

$$\pi(u) \leq \pi(\lfloor u/v \rfloor \, v) + \pi(\gamma(u)) \leq \lfloor u/v \rfloor \, \pi(v) + \pi(\gamma(u)) .$$

Hence, by $\pi(v)/|v| \leq \alpha$,

$$\pi(u) \leq \lfloor u/v \rfloor \, \alpha |v| + \pi(\gamma(u)) \leq \alpha |u| + \pi(\gamma(u)) .$$

Since the limsup exists,

$$\pi(\gamma(u)) \leq (\beta + \delta) \, |\gamma(u)| \leq (\beta + \delta) \, |v|$$

for some $\delta > 0$, provided v is small enough. Hence

$$\pi(u) \leq \alpha |u| + (\beta + \delta) \, |v| ,$$

and as $v \to 0$, we have $\pi(u)/|u| \leq \alpha$, a contradiction to $\pi(u)/|u| > \alpha$. Thus the lemma is proven.

Clearly, we have the same property for $\limsup \{\pi(u)/(1 - |u|)\colon u \uparrow 1\}$ and $\lim \{\pi(u)/(1 - |u|)\colon u \uparrow 1\}$.

Lemma 2.4. *If π is a subadditive function on I and if*

$$\lim_{u \downarrow 0} \{\pi(u)/|u|\} = \beta ,$$

then

$$\limsup_{u \downarrow v} \{(\pi(u) - \pi(v))/(|u| - |v|)\} \leq \beta$$

for any $v \in I$.

Proof. By subadditivity, $\pi(u) \leq \pi(v) + \pi(u - v)$. By $\beta = \lim \{\pi(u)/|u|: u \downarrow 0\}$ for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$\pi(u - v) \leq (\beta + \epsilon)(|u| - |v|)$$

for $|u| > |v|$ and $|u| - |v| \leq \delta$. For such $u, v \in I$,

$$\pi(u) \leq \pi(v) + (\beta + \epsilon)(|u| - |v|),$$

or

$$(\pi(u) - \pi(v))/(|u| - |v|) \leq \beta + \epsilon .$$

The lemma is thus proven.

Clearly a similar statement holds for $\lim \{\pi(u)/(1 - |u|): u \uparrow 1\}$ and

$$\limsup \{(\pi(u) - \pi(v))/(|v| - |u|): u \uparrow v\} .$$

Property 2.5. *If $\pi' = (\pi, \pi^+, \pi^-)$ is a minimal valid inequality for* $P_-^+(I, u_0)$, *then*

$$\pi^+ = \lim_{u \downarrow 0} \{\pi(u)/|u|\} , \quad \pi^- = \lim_{u \uparrow 1} \{\pi(u)/(1 - |u|)\}$$

Proof. By subadditivity of π', $\pi(u) \leq \pi^+ |u|$, so

$$\limsup_{u \downarrow 0} \{\pi(u)/|u|\} \leq \pi^+ .$$

Then Lemma 2.3 implies that $\lim \{\pi(u)/|u|: u \downarrow 0\}$ exists and is less than or equal to π^+. Similarly, $\lim \{\pi(u)/(1 - |u|): u \uparrow 1\}$ exists and is less than or equal to π^-. If either the limit is less than π^+ or π^-, respectively, then Property 2.2 implies that π' is not a minimal valid inequality, and the proof is complete.

Property 2.6. *If π is a subadditive function on I and if $\pi(u) \to 0$ as $u \downarrow 0$ and $\pi(u) \to 0$ as $u \uparrow 1$, then π is continuous at every $u \in I$.*

Proof. For any $u \in I$,

$$\pi(u+\delta) - \pi(\delta) \leq \pi(u) \leq \pi(u+\delta) + \pi(-\delta)\,,$$

or

$$-\pi(\delta) \leq \pi(u) - \pi(u+\delta) \leq \pi(-\delta)\,.$$

As $\delta \downarrow 0$, we have $-\delta \uparrow 1$ (since δ is a group element and $-\delta = 1 - \delta$), and $u + \delta \downarrow u$. Therefore, $\pi(u+\delta) \to \pi(u)$ as $u + \delta \downarrow u$. Now, letting $\delta \uparrow 1$ gives $-\delta \downarrow 0$ and $u + \delta \uparrow u$, so that $\pi(u+\delta) \to \pi(u)$ as $u + \delta \uparrow u$.

Theorem 1.6 applies here, since I is trivially a subgroup of itself, and says that a valid inequality π for $\mathrm{P}(I, u_0)$ is minimal if and only if $\pi(u) + \pi(u_0 - u) = 1$ for all $u \in I$. The analogous results for $\mathrm{P}^{+}_{-}(I, u_0)$ will now be given.

Property 2.7. A valid inequality $\pi' = (\pi, \pi^+, \pi^-)$ for $\mathrm{P}^{+}_{-}(I, u_0)$ is minimal if and only if

$$\pi(u) + \pi(u_0 - u) = 1\,, \qquad u \in I\,, \tag{2.3}$$

$$\pi^+ = \lim_{u \downarrow 0} \{\pi(u)/|u|\}\,, \tag{2.4}$$

$$\pi^- = \lim_{u \uparrow 1} \{\pi(u)/(1 - |u|)\}\,. \tag{2.5}$$

Proof. Suppose π' is a minimal valid inequality. Then by Property 2.5, (2.4) and (2.5) hold. Furthermore, Property 2.1 implies that π is a valid inequality for $\mathrm{P}(I, u_0)$. If π is not a minimal valid inequality for $\mathrm{P}(I, u_0)$, then there is a valid inequality $\rho < \pi$, and (ρ, π^+, π^-) is a valid inequality for $\mathrm{P}^{+}_{-}(I, u_0)$ by

$$\begin{aligned}
&\sum_{u \in I} \rho(u)\, t(u) + \pi^+ s^+ + \pi^- s^- \\
&\quad\geq \sum_{u \in I} \rho(u)\, t(u) + \pi(\mathcal{F}(s^+)) + \pi(\mathcal{F}(-s^-)) \\
&\quad\geq \sum_{u \in I} \rho(u)\, t(u) + \rho(\mathcal{F}(s^+)) + \rho(\mathcal{F}(-s^-)) \geq 1\,,
\end{aligned}$$

since ρ is a valid inequality. We can use $\pi^+ s^+ \geq \pi(\mathcal{F}(s^+))$, and similarly for $\pi^- s^-$, because π' is minimal and hence subadditive by Theorem 1.2. Therefore, π must be minimal, and (2.3) must hold.

We have shown that if π' is a minimal valid inequality for $P^{+}_{-}(I, u_0)$, then (2.3)–(2.5) must hold. As a corollary, we have seen that π must be a minimal valid inequality for $P(I, u_0)$.

Now suppose (2.3)–(2.5) hold for a valid inequality π' for $P^{+}_{-}(I, u_0)$. If $\rho' < \pi'$ for $\rho' = (\rho, \rho^+, \rho^-)$ a valid inequality for $P^{+}_{-}(I, u_0)$, then at least one of $\rho^+ < \pi^+$, $\rho^- < \pi^-$ or $\rho(u) < \pi(u)$ for some $u \in I$ must hold. The latter possibility is ruled out by (2.3), just as in the proof of Theorem 1.6 in [4]. Hence $\rho(u) = \pi(u)$ for all $u \in I$. Hence at least one of $\rho^+ < \pi^+$, $\rho^- < \pi^-$ must hold. We will reach a contradiction by supposing $\rho^+ < \pi^+$, and the proof is similar if $\rho^- < \pi^-$.

Suppose $\rho^+ < \pi^+$. By (2.4), there is some $v \in I$ with $\rho^+ < \pi(v)/|v|$, and, hence, $\rho^+ |v| < \pi(v)$. But then $t(u_0 - v) = 1$, $s^+ = |v|$ is a solution for $P^{+}_{-}(I, u_0)$ satisfying

$$\rho(u_0 - v)\, t(u_0 - v) + \rho^+ s^+ = \pi(u_0 - v) + \rho^+ |v| < \pi(u_0 - v) + \pi(v) = 1$$

by (2.3). Hence ρ' is not a valid inequality for $P^{+}_{-}(I, u_0)$, completing the proof.

Property 2.8. *If π is an extreme valid inequality for* $P(I, u_0)$ *and* π^+, π^- *are given by* (2.4) *and* (2.5), *then* $\pi' = (\pi, \pi^+, \pi^-)$ *is an extreme valid inequality for* $P^{+}_{-}(I, u_0)$.

Proof. By Property 2.2, π' is a valid inequality since we are assuming the existence of the limits in (2.4) and (2.5). By Theorem 1.1, π is minimal, so (2.3) holds. Hence, by the previous property, π' is a minimal valid inequality for $P^{+}_{-}(I, u_0)$.

Suppose π' is not extreme. Then there are valid inequalities ρ' and σ' for $P^{+}_{-}(I, u_0)$ with

$$\pi' = \tfrac{1}{2}\rho' + \tfrac{1}{2}\sigma' = \tfrac{1}{2}(\rho, \rho^+, \rho^-) + \tfrac{1}{2}(\sigma, \sigma^+, \sigma^-)\,.$$

Now, ρ' and σ' must both be minimal by [4, Lemma 1.4] since π' is minimal. By Property 2.1, ρ and σ are valid inequalities for $P(I, u_0)$. By hypothesis, π is an extreme valid inequality for $P(I, u_0)$, so $\rho = \sigma = \pi$. By ρ' and σ' being minimal valid inequalities for $P^{+}_{-}(I, u_0)$ and by Property 2.5, $\rho^+ = \sigma^+ = \pi^+$ and $\rho^- = \sigma^- = \pi^-$ because $\rho = \sigma = \pi$. Thus, π' is extreme.

Property 2.9. *If* $\pi' = (\pi, \pi^+, \pi^-)$ *is an extreme valid inequality for* $P^+_-(I, u_0)$, *then* π *is an extreme valid inequality for* $P(I, u_0)$.

Proof. Since π' is extreme, it is also minimal, and by Properties 2.1 and 2.7, π is a minimal valid inequality for $P(I, u_0)$. It is, therefore, a subadditive function on I by Theorems 1.2 and 1.4. Suppose π is not an extreme valid inequality for $P(I, u_0)$. Then

$$\pi = \tfrac{1}{2}\rho + \tfrac{1}{2}\sigma\,, \quad \rho \neq \sigma\,,$$

where ρ and σ must be minimal valid inequalities for $P(I, u_0)$ by [4, Lemma 1.4]. Then $\frac{1}{2}\rho \leq \pi$ and

$$\limsup_{u \downarrow 0} \{\rho(u)/|u|\} \leq \limsup_{u \downarrow 0} \{2\pi(u)/|u|\} = 2\pi^+\,.$$

Hence $\lim\{\rho(u)/|u|\colon u \downarrow 0\}$ exists by Lemma 2.3; call it l_1^+. Similarly, $\lim\{\rho(u)/(1 - |u|)\colon u \uparrow 1\}$ exists; let us call it l_1^-. Obviously, the same limits exists for σ; let us call them l_2^+ and l_2^-. By Property 2.5 and by $\pi = \frac{1}{2}\rho + \frac{1}{2}\rho$, it follows that $\pi^+ = \frac{1}{2}l_1^+ + \frac{1}{2}l_2^+$ and $\pi^- = \frac{1}{2}l_1^- + \frac{1}{2}l_2^-$. Hence $\rho' = (\pi, l_1^+, l_1^-)$ and $\sigma' = (\sigma, l_2^+, l_2^-)$ are valid inequalities for $P^+_-(I, u_0)$ by Property 2.2. But $\pi' = \frac{1}{2}\rho' + \frac{1}{2}\sigma'$, which is a contradiction to π' being extreme. Thus the property is proven.

These nine results give a fairly complete picture of the relation between extreme valid inequalities for the two problems $P^+_-(I, u_0)$ and $P(I, u_0)$. In addition, the results give some idea as to what these extreme valid inequalities are like.

3. Extreme valid inequalities

3.1. *Construction of some extreme inequalities for* $P^+_-(U, u_0)$

We will see how to construct some extreme valid inequalities for $P(I, u_0)$ and $P^+_-(I, u_0)$ from extreme valid inequalities for $P^+_-(G_n, u_0)$. Let $\pi' = (\pi, \pi^+, \pi^-)$ be the valid inequality for $P^+_-(I, u_0)$ obtained by a two-slope fill-in of an extreme valid inequality for $P^+_-(G_n, u_0)$.

Theorem 3.1. *π' is an extreme valid inequality for* $P^+_-(U, u_0)$ *for any subset U of I which contains G_n and for which*

$$\pi(u) + \pi(u_0 - u) = 1 \ , \ u \in U \ .$$

Proof. We know by Theorem 1.10 that π' is a valid inequality for $P^+_-(U, u_0)$.

(i) We first show that it is also a minimal valid inequality for $P^+_-(U, u_0)$. Suppose it is not minimal. Then there is a valid inequality ρ' for $P^+_-(U, u_0)$ with $\rho' < \pi'$. By the construction of π', it is an extreme valid inequality for $P^+_-(G_n, u_0)$, and hence π' is a minimal valid inequality for $P^+_-(G_n, u_0)$. Since ρ' is a valid inequality for $P^+_-(G_n, u_0)$ because $G_n \subseteq S$, we must have $\rho' = \pi'$ on G_n, and $\rho^+ = \pi^+$, $\rho^- = \pi^-$ as well. Hence $\rho(v) < \pi(v)$ for some $v \in U \backslash G_n$. By the construction of π', for the complementary point $u_0 - v$,

$$\begin{aligned}\pi(u_0 - v) = \min \{&\pi(L(u_0 - v)) + \pi^+(|u_0 - v| - |L(u_0 - v)|) \ , \\ &\pi(R(u_0 - v) + \pi^-(|R(u_0 - v)| - |u_0 - v|)\} \ .\end{aligned}$$

Suppose the first term in brackets gives $\pi(u_0 - v)$. Then $s^+ = |u_0 - v| - |L(u_0 - v)|$, $t(v) = 1$, $t(L(u_0 - v)) = 1$ is a solution to $P^+_-(S, u_0)$, but

$$\begin{aligned}&\sum_{u \in S} \rho(u)\, t(u) + \rho^+ s^+ + \rho^- s^- \\ &\quad = \rho(v) + \pi(L(u_0 - v) + \pi^+(|u_0 - v| - |L(u_0 - v)|) \\ &\quad = \pi(u_0 - v) + \rho(v)\ \pi(u_0 - v) + \pi(v) = 1 \ ,\end{aligned}$$

contradicting ρ' being a valid inequality for $P^+_-(U, u_0)$. When $\pi(u_0 - v)$ is equal to the second term in brackets, the proof is similar but uses the solution $s^- = |R(u_0 - v)| - |u_0 - v|$, $t(R(u_0 - v)) = 1$, $t(v) = 1$.

(ii) Next we show that π' is extreme among the subadditive valid inequalities for $P^+_-(U, u_0)$. This result, together with minimality, will show that π' is an extreme valid inequality for $P^+_-(U, u_0)$ by Theorem 1.3.

Suppose π' is not an extreme subadditive valid inequality. Then $\pi' = \frac{1}{2}\rho' + \frac{1}{2}\sigma'$ for subadditive valid inequalities ρ' and σ'. Just as in the proof of minimality, π' is an extreme valid inequality for $P^+_-(G_n, u_0)$, so

$$\pi^+ = \rho^+ = \sigma^+ , \qquad \pi^- = \rho^- = \sigma^- ,$$

$$\pi(g_i) = \rho(g_i) = \sigma(g_i) ,$$

and hence $\rho' \neq \sigma'$ means that $\rho(v) \neq \sigma(v)$ for some $v \in U \backslash G_n$. Since $\pi(v) = \frac{1}{2}\rho(v) + \frac{1}{2}\sigma(v)$, one of $\rho(v)$, $\sigma(v)$ is larger than $\pi(v)$ and one is smaller. Without loss of generality, we can assume $\rho(v) > \pi(v) > \sigma(v)$. Again, by the construction of $\pi(v)$, we have either

$$\pi(v) = \pi(L(v)) + \pi^+(|v| - |L(v)|)$$

or

$$\pi(v) = \pi(R(v)) + \pi^-(|R(v)| - |v|) .$$

Let us assume that $\pi(v)$ is given by the first expression, and the proof in the second case is similar. By the subadditivity of ρ' and by $\rho^+ = \pi^+$,

$$\rho(L(v)) + \pi^+(|v| - |L(v)|) \geq \rho(v) .$$

But $\rho(L(v)) = \pi(L(v))$ since $L(v) \in G_n$. Hence

$$\pi(L(v)) + \pi^+(|v| - |L(v)|) \geq \rho(v) .$$

But here the left-hand side is equal to $\pi(v)$ by our assumption of case (i) above. Hence $\pi(v) \geq \rho(v)$, contradicting $\rho(v) > \pi(v)$. The proof is thus completed.

We can apply this theorem to Table 2 of the appendix of [4]. Corresponding to each extreme valid inequality for $\mathrm{P}_-^+(G_n, u_0)$, $n = 1, ..., 6$, we can easily give the set U_c on which $\pi(u) + \pi(u_0 - u) = 1$, $u \in S_\mathrm{c}$. Then for any set U, $G_n \subset U \subset U_\mathrm{c}$, the inequality given by Theorem 1.10 is an extreme valid inequality for $\mathrm{P}_-^+(U, u_0)$. For G_0, G_1, G_2, G_3, G_4 and G_6, $U_\mathrm{c} = I$ for all extreme valid inequalities of $\mathrm{P}_-^+(G_n, u_0)$; the first exception occurs at G_5. There are four exceptions for G_5 among the 6 faces given by [4, Table 2] and the reflections. These exceptions are discussed further following Corollary 4.4.

The unique extreme valid inequality for $\mathrm{P}_-^+(G_0, u_0)$, where G_0 is the subset consisting of only the point 0, is of particular interest. It is readily seen that this inequality π' when used in conjunction with a mapping φ gives Gomory's mixed integer cut [1, p. 528]. We see at once that for this π', $\pi(u) + \pi(u_0 - u) = 1$ for all u so that π' is an extreme valid inequality for $P_-^+(U, u_0)$ for any set $S \subset I$ provided $0 \in S$, which is actually not a restriction since 0 can always be adjoined to S without changing the problem.

3.2. *Some extreme inequalities for* $P(G_m, u_0)$

When the inequalities π' given by the two-slope fill-in (Theorem 1.10) satisfy $\pi(u) + \pi(u_0 - u) = 1$, then the theorem just proven says that π' is an extreme valid inequality for $P_-^+(I, u_0)$. By Property 2.9, π is an extreme valid inequality for $P(I, u_0)$. For subsets U of I, we know that π is a valid inequality for $P(U, u_0)$, but we do not know that π is extreme for $P(U, u_0)$. The following theorem establishes that result for some U and, in fact, applies for any extreme valid inequality for $P(I, u_0)$, not just those given by the two-slope fill-in.

Theorem 3.2. *If π is an extreme valid inequality for* $P(I, u_0)$ *and consists of straight line segments connected at values u belonging to a regular grid G_m with $u_0 \in G_m$, then π is an extreme valid inequality for* $P(U, u_0)$ *whenever U is a subset of I including G_m.*

Proof. Since π is extreme for $P(I, u_0)$, it cannot be written as $\frac{1}{2}\rho + \frac{1}{2}\sigma$ for different valid inequalities ρ, σ for $P(I, u_0)$. Certainly π is a valid inequality for $P(U, u_0)$, and if it is not extreme for $P(U, u_0)$, then $\pi = \frac{1}{2}\rho + \frac{1}{2}\sigma$ for different valid inequalities ρ, σ for $P(U, u_0)$. If both ρ and σ are valid inequalities for $P(I, u_0)$, a contradiction is reached. However, both can be extended to valid inequalities for $P(I, u_0)$ by the straight-line fill-in from G_m as in Theorem 1.10. Furthermore, such a construction maintains $\pi = \frac{1}{2}\rho + \frac{1}{2}\sigma$ on all of I since π also consists of straight line segments joined at points of G_m. The proof is thus completed.

This theorem enables us to construct some extreme valid inequalities (faces) of the polyhedra $P(G, g_0)$ of [3]. It is of particular interest when one extreme inequality of $P_-^+(G_n, u_0)$ gives rise to many slight variants, all of which are extreme for $P(I, u_0)$ and all of which in turn give rise to apparently unrelated faces of $P(G, u_0)$. Before showing that possibliity, we digress to give some results related to the two-slope construction of Theorem 1.10.

3.3. *Extremality of two-slope functions*

Theorem 3.3. *Let π be a continuous function on I consisting of a finite number of straight line segments, each line segment having a slope $\pi^+ > 0$ or else $-\pi^- < 0$. If π is a subadditive function on I with $\pi(u_0) = 1$ for some $u_0 \in I$, then π is extreme among the subadditive valid inequalities ρ for* P(I, u_0) *which have $\rho(u_0) = 1$.*

Proof. The theorem asserts that if $\pi = \frac{1}{2}\rho + \frac{1}{2}\sigma$, where ρ and σ are subadditive valid inequalities for P(I, u_0) with $\rho(u_0) = \sigma(u_0) = 1$, then $\rho(u) = \sigma(u)$ for all $u \in I$. We know from Theorem 1.4 that π is a subadditive valid inequality for P(I, u_0).

Suppose that $\pi = \frac{1}{2}\rho + \frac{1}{2}\sigma$ for subadditive valid inequalities ρ, σ for P(I, u_0) with $\rho(u_0) = \sigma(u_0) = 1$. Since π has a right-hand derivative π^+ at 0,

$$\limsup_{u \downarrow 0} \{\rho(u)/|u|\} \leq \limsup_{u \downarrow 0} \{2\pi(u)/|u|\} = 2\pi^+ ,$$

and similarly for σ. By Lemma 2.3, ρ and σ both have right-hand derivatives ρ^+ and σ^+ at 0. Similarly, the left-hand derivatives ρ^- and σ^- at 1 exist.

We next show that ρ and σ have the same form as π; that is, continuous line segments of slope ρ^+ or ρ^- (σ^+ or σ^-). Choose a point u within an interval where π has slope π^+. Let $\delta > 0$ be small enough that $u + \delta$ is in the same interval and that δ' itself lies in the very first interval. Then, $\pi(u) + \pi(\delta) = \pi(u + \delta)$ by the fact that π has the same slope π^+ on $(0, \delta)$ and $(u, u + \delta)$. Hence

$$\tfrac{1}{2}\rho(u) + \tfrac{1}{2}\sigma(u) + \tfrac{1}{2}\rho(\delta) + \tfrac{1}{2}\sigma(\delta) = \tfrac{1}{2}\rho(u + \delta) + \tfrac{1}{2}\sigma(u + \delta) ,$$

or

$$\tfrac{1}{2}(\rho(u) + \rho(\delta) - \rho(u + \delta)) + \tfrac{1}{2}(\sigma(u) + \sigma(\delta) - \sigma(u + \delta)) = 0 .$$

By subadditivity, each of $\rho(u) + \rho(\delta) - \rho(u + \delta)$ and $\sigma(u) + \sigma(\delta) - \sigma(u + \delta)$ is non-negative. Since they sum to zero, each must be zero. Hence

$$\lim_{\delta \downarrow 0} \{(\rho(u + \delta) - \rho(u))/|\delta|\} = \lim_{\delta \downarrow 0} \{\rho(\delta)/|\delta|\} = \rho^+ ,$$

$$\lim_{\delta \downarrow 0} \{(\sigma(u + \delta) - \sigma(u))/|\delta|\} = \lim_{\delta \downarrow 0} \{\sigma(\delta)/|\delta|\} = \sigma^+ .$$

Similarly, we can show that the left-hand derivatives of ρ and σ at u are ρ^+ and σ^+. Therefore, ρ (resp. σ) has a constant derivative ρ^+ (resp. σ^+) on the interval, and so it is a straight line with this slope. A similar result is obtained for any x on an interval where π has slope $-\pi^-$. Here one works with subadditivity through the inequality $\rho(-\delta) + \rho(u + \delta) \geq \rho(u)$, and concludes that both the left and right derivatives at u are ρ^-. Hence both ρ and σ are of the same form as π with two slope straight line segments over the same intervals.

We now show that $\rho^+ = \sigma^+ = \pi^+$ and $\rho^- = \sigma^- = \pi^-$. Let l_L^+ be the total length of the intervals on which the slope of π is π^+ and which lie to the left of u_0. Similarly, let l_R^+ be the length of those intervals to the right of u_0 on which π has slope π^+, and let l_L^- and l_R^- be the corresponding lengths of intervals on which π has slope $-\pi^-$. Since $\pi(u_0) = 1$,

$$\pi^+ \, l_L^+ - \pi^- \, l_L^- = 1 \,, \qquad \pi^+ \, l_R^+ - \pi^- \, l_R^- = -1 \,,$$

and the same equations hold for ρ^+, ρ^- and σ^+, σ^-. But these two equations have only the solution π^+, π^- because in order for them to have more than one solution, one equation would have to be a linear multiple of the other. But then $l_L^+ + l_R^+ = 0$ and $-l_L^- - l_R^- = 0$, implying that all of l_L^+, l_R^+, l_L^- and l_R^- are zero. Hence $\rho^+ = \pi^+$, $\rho^- = \pi^-$, and $\sigma^+ = \pi^+$, $\sigma^- = \pi^-$.

We have two immediate corrolaries.

Corollary 3.4. *If π meets the conditions of Theorem* 1.10 *and if $\pi(u) + \pi(u_0 - u) = 1$ for all $u \in I$, then π is an extreme valid inequality for* $\mathrm{P}(I, u_0)$.

Proof. If $\pi(u) + \pi(u_0 - u) = 1$ for all $u \in I$, then by Theorem 1.6, π is a minimal valid inequality for $\mathrm{P}(I, u_0)$. The subadditive valid inequalities ρ for which $\rho(u_0) = 1$ include the minimal valid inequalities by Theorems 1.2 and 1.6. Since by Theorem 1.10, π is extreme among those inequalities, π cannot be written as a mid-point of two other minimal valid inequalities. By [4, Lemma 1.4] and the minimality of π, π is an extreme valid inequality for $\mathrm{P}(I, u_0)$.

Corollary 3.5. *If π meets the conditions of Theorem* 1.10, *if $\pi(u) + \pi(u_0 - u) = 1$ for all $u \in I$, and if π^+, $-\pi^-$ are the two slopes of π with $\pi^+ > 0$, $\pi^- > 0$, then $\pi' = (\pi, \pi^+, \pi^-)$ is an extreme valid inequality for* $\mathrm{P}_-^+(I, u_0)$.

Proof. This is immediate from Corollary 3.4 and Property 2.8.

Before leaving this section, let us point out the difference between Theorem 3.1 and Corollary 3.4. Corollary 3.4 and Theorem 3.2 would prove that π of Theorem 3.1 is extreme when U includes the cyclic group including all of the break points of π, not just G_n. However, Theorem 3.1 applies only to those π constructed using the two-slope fill-in, whereas Corollary 3.4 applies to arbitrary two-slope functions which are subadditive and minimal.

4. Generating extreme inequalities and exponential growth for faces of some $\mathrm{P}(G, u_0)$

We begin by discussing some of the possibilities for creating extreme inequalities for $\mathrm{P}(I, u_0)$ from extreme inequalities of $\mathrm{P}_-^+(G_n, u_0)$ when the condition $\pi(u) + \pi(u_0 - u) = 1$ does not hold for all $u \in I$ for the π constructed by the two-slope fill-in.

By way of background, we observe that the π given by the two-slope fill-in of Theorem 1.10 does satisfy $\pi(u) + \pi(u_0 - u) = 1$ when $u \in G_n$. This fact is a consequence of (1.20)–(1.22) because they imply

$$\begin{aligned}\pi(g_i) &= \min\{\pi(L(u_0)) - \pi(L(u_0) - g_i)\,,\\ &\qquad \pi(R(u_0)) - \pi(R(u_0) - g_i)\}\\ &= \min\{1 - \pi^+(|u_0 - L(u_0)|) - \pi(L(u_0) - g_i)\,,\\ &\qquad 1 - \pi^-(|R(u_0) - u_0|) - \pi(R(u_0) - g_i)\}\,.\end{aligned}$$

Hence

$$\begin{aligned}&\pi(g_i) + \min\{\pi(L(u_0) - g_i) + \pi^+(|u_0 - L(u_0)|)\,,\\ &\qquad \pi(R(u_0) - g_i) + \pi^-(|R(u_0) - u_0|)\} = 1\,.\end{aligned}$$

By the construction of π on $I \backslash G_n$ and by $L(u_0 - g_i) = L(u_0) - g_i$ and $R(u_0 - g_i) = R(u_0) - g_i$, the minimum in the equation above is precisely $\pi(u_0 - g_i)$. Since $\pi(u) + \pi(u_0 - u) = 1$ for $u \in G_n$, equality also clearly holds for $u = u_0 - g_i$, $g_i \in G_n$. These points are located between consecutive grid points $L(u)$, $R(u)$ in the same relative position as u_0 is between $L(u_0)$ and $R(u_0)$.

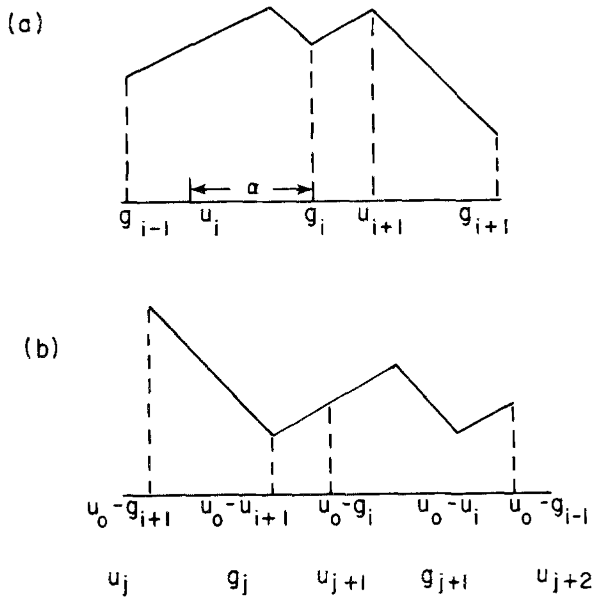

Fig. 1.

Figs. 1(a) and (b) illustrate the possibilities for π on the intervals g_{i-1}, g_i, g_{i+1} and the complementary intervals $u_0 - u_{i+2}$, $u_0 - u_{i+1}$, $u_0 - u_i$, $u_0 - u_{i-1}$, where we let $u_{i+1} = g_i + u_0 - L(u_0)$ and $u_i = g_i - R(u_0) + u_0$. Then $u_0 - u_{i+1} \in G_n$, say $g_j = u_0 - u_{i+1}$ and $g_{j+1} = u_0 - u_i \in G_n$. If, as in Fig. 1, the maximum of π in (g_i, g_{i+1}) occurs at $u = u_{i+1}$, then $\pi(u) + \pi(u_0 - u) = 1$ for all $u \in (g_i, g_{i+1})$. In order to see this result, consider any interval (u_{i+1}, g_{i+1}) where $u_{i+1} = g_i + u_0 - L(u_0) = g_{i+1} - R(u_0) + u_0$, and the complementary interval $(u_0 - g_{i+1}, u_0 - u_{i+1})$. The difference $\pi(u_{i+1}) - \pi(g_{i+1})$ must be the same as $\pi(u_0 - g_{i+1}) - \pi(u_0 - u_{i+1})$ because $\pi(g_{i+1}) + \pi(u_0 - g_{i+1}) = 1$ and $\pi(u_i) + \pi(u_0 - u_i) = 1$. Since π can have only two slopes, π must be the same, except for a constant difference in height, in the two intervals (u_{i+1}, g_{i+1}) and $(u_0 - g_{i+1}, u_0 - u_{i+1})$.

The second possibility is illustrated in Fig. 1 by the interval (u_i, g_i) and its complementary interval $(u_0 - g_i, u_0 - u_i)$. In both intervals, π has two slopes and a relative maximum occurs within the interval, In this case, we must have $\pi(u) + \pi(u_0 - u) > 1$ for all u within either interval. For at $u = u_i$, $\pi(u) + \pi(u_0 - u) = 1$, but as u is increased, both $\pi(u)$ and $\pi(u_0 - u)$ increase until one of $\pi(u)$, $\pi(u_0 - u)$ reaches a maxima. Then $\pi(u) + \pi(u_0 - u)$ remains constant as u increases since one of

$\pi(u)$, $\pi(u_0 - u)$ is increasing while the other is decreasing at the same rate. When the other $\pi(u)$, $\pi(u_0 - u)$ reaches its maxima, then $\pi(u) + \pi(u_0 - u)$ decreases until u reaches g_i and $u_0 - u$ reached $u_0 - g_i$ at which point $\pi(u) + \pi(u_0 - u) = 1$.

An interval (u_i, g_i) or (g_i, u_{i+1}) with only one slope for π will be called an *interval of the first type*; here, $u_{i+1} = g_i + u_0 - L(u_0)$. The complementary interval will also be an interval of the first type, and for u in an interval of the first type, $\pi(u) + \pi(u_0 - u) = 1$. An interval (u_{i-1}, g_i) or (g_i, u_{i+1}) with two slopes for π will be called an *interval of the second type.* Then its complementary interval is also of the second type, and for u within an interval of the second type, $\pi(u) + \pi(u_0 - u) > 1$. We note that the intervals $(L(u_0), u_0)$ and $(u_0, R(u_0))$ are of the first type, and so are their complementary intervals $(0, u_0 - L(u_0))$, $(1 - R(u_0) + u_0, 1)$.

An interval (u_i, g_i) will be its own complement if $g_i + g_i = R(u_0)$ since then $u_0 - g_i = u_0 - R(u_0) + g_i = u_i$. The interval (g_i, u_{i+1}) will be its own complement if $g_i + g_i = L(u_0)$ since then $u_0 - g_i = u_0 - L(u_0) + g_i = u_{i+1}$. These self-complementary intervals may be of either the first or second type. In what follows, we will exclude the self-complementary intervals in the discussion of intervals of the second type.

With this background, we can construct a function π_α from π which will lead to some interesting results. Let $\alpha = (g_i, u_{i+1})$ be an interval of the second type and let β be its complementary interval. We assume that α is not its own complement, so $\alpha \neq \beta$. Then $\pi(u) + \pi(u_0 - u) > 1$ for u within either α or β. Define π_α on I by

$$\pi_\alpha(u) = \begin{cases} \pi(u)\,, & u \in I \backslash \alpha\,, \\ 1 - \pi(u_0 - u)\,, & u \in \alpha\,. \end{cases}$$

Fig. 2 illustrates π_α in this case. Let u_α denote the u where $\pi_\alpha(u)$ is smallest in α.

First, two lemmas are needed. The first applies to any π and does not depend on the particular construction here.

Lemma 4.1. *Let S be a subset of I and let π be a subadditive valid inequality for* $\mathrm{P}(S, u_0)$. *If*

$$\pi(u) + \pi(u_0 - u) \geq 1\,, \qquad u \in I \backslash S\,,$$
$$\pi(u) + \pi(v) \geq \pi(u + v)\,, \quad u \in I \backslash S\,, \quad v \in I \backslash S\,,$$

then π is a valid inequality for $\mathrm{P}(I, u_0)$.

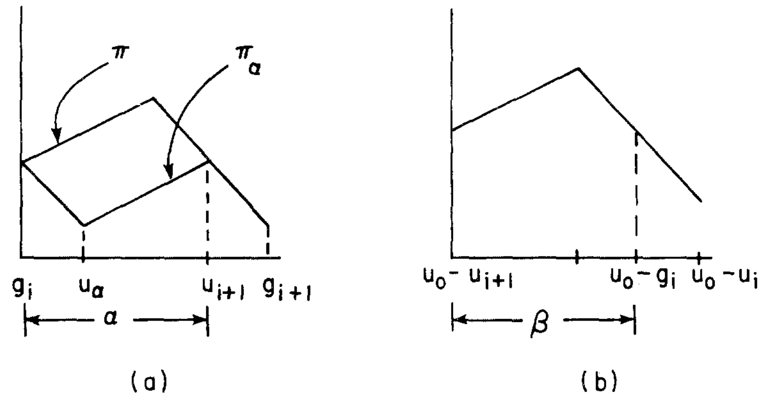

Fig. 2.

Proof. Consider any t solving $\mathrm{P}(I, u_0)$. If $t(u) > 0$ and $t(v) > 0$ for both u and v in $I \backslash S$, then we can change t by reducing $t(u)$ by 1, reducing $t(v)$ by 1, and increasing $t(u + v)$ by 1. The new t is still a solution, and since $\pi(u) + \pi(v) \geq \pi(u + v)$, $\Sigma_{u \in I} \pi(u)\, t(u)$ has not increased. This process can be continued until $\Sigma_{u \in I \backslash S} t(u) \leq 1$. At that point,

$$\sum_{u \in I} \pi(u)\, t(u) = \pi(v) + \sum_{u \in S} \pi(u)\, t(u) \,,$$

where $v \in I \backslash S$. By subadditivity of π on S,

$$\sum_{u \in I} \pi(u)\, t(u) \geq \pi(v) + \pi \left(\sum_{u \in S} u\, t(u) \right)$$

$$= \pi(v) + \pi(u_0 - v) \geq 1 \,,$$

by $\pi(u) + \pi(u_0 - u) \geq 1$ for $u \in I \backslash S$. The lemma is therefore proven.

The second lemma applies to the particular function π_α constructed here. It actually applies to any two-slope function π in an interval in which the function first decreases and then increases.

Lemma 4.2. *If* $2\pi_\alpha(u_\alpha) \geq \pi(2u_\alpha)$, *then* $\pi_\alpha(u) + \pi_\alpha(v) \geq \pi_\alpha(u + v)$ *for all* $u, v \in \alpha$.

Proof. For any $u \in \alpha$, $u \neq u_\alpha$, either $|u| < |u_\alpha|$ or $|u| > |u_\alpha|$. Let us assume $|u| > |u_\alpha|$. The other case is similar. Then

$$\pi_\alpha(u) = \pi_\alpha(u_\alpha) + \pi^+(|u| - |u_\alpha|)$$

and for $v \in \alpha$,

$$\begin{aligned}\nabla_\alpha(u, v) &= \pi_\alpha(u) + \pi_\alpha(v) - \pi_\alpha(u+v)\\ &= \pi_\alpha(u_\alpha) + \pi^+(|u| - |u_\alpha|) + \pi_\alpha(v) - \pi_\alpha(u+v)\\ &= \pi_\alpha(u_\alpha) + \pi_\alpha(v) - (\pi_\alpha(u+v) - \pi^+(u+v| - |u_\alpha + v|))\\ &\geq \pi_\alpha(u_\alpha) + \pi_\alpha(v) - \pi_\alpha(u_\alpha + v) = \nabla_\alpha(u_\alpha, v)\,,\end{aligned}$$

by

$$\pi_\alpha(u_\alpha + v) + \pi^+(|u+v| - |u_\alpha + v|) \geq \pi_\alpha(u_\alpha + v)\,.$$

Similarly, we can show $\nabla_\alpha(u_\alpha, v) \geq \nabla_\alpha(u_\alpha, u_\alpha)$. Hence if $\nabla_\alpha(u_\alpha, u_\alpha) \geq 0$, then $\nabla_\alpha(u, v) \geq 0$ for all $u, v \in \alpha$.

These two lemmas suffice to prove the following theorem.

Theorem 4.3. *If* $2\pi_\alpha(u_\alpha) \geq \pi_\alpha(2u_\alpha)$, *then* π_α *is a valid inequality for* $\mathrm{P}(I, u_0)$.

Proof. By Lemma 4.1, we need only show that $\pi_\alpha(u) + \pi_\alpha(u_0 - u) \geq 1$ for all $u \in \alpha$ and $\pi(u) + \pi(v) \geq \pi(u+v)$ for all $u, v \in \alpha$. The first inequality is true, in fact with equality, by the construction of π_α. The second is true by $2\pi_\alpha(u_\alpha) \geq \pi_\alpha(2u_\alpha)$ and Lemma 4.2.

Corollary 4.4. *If* α *and its complement* β *are the only two intervals of the second type, then* π_α *is an extreme valid inequality for* $\mathrm{P}(I, u_0)$ *if and only if* $2\pi_\alpha(u_\alpha) \geq \pi(2u_\alpha)$.

Proof. If $2\pi_\alpha(u_\alpha) \geq \pi_\alpha(2u_\alpha)$, then by Theorem 4.3, π_α is a valid inequality for $\mathrm{P}(I, u_0)$. Furthermore, if α and β are the only two intervals of the second type, then $\pi_\alpha(u) + \pi_\alpha(u_0 - u) = 1$ for all $u \in I$, so π_α is minimal. By Corollary 3.4, π_α is an extreme valid inequality for $\mathrm{P}(I, u_0)$.

We now consider in more detail the case described in Corollary 4.4. To begin, two cases will be shown from [4, Table 2]. When $n = 5$ and $u_0 \in (0, \frac{1}{5})$, face 2 from [4, Table 2] is illustrated in Fig. 3. Of course, when $u_0 \in (\frac{4}{5}, 1)$, the reflection is also a face of $\mathrm{P}^+_-(G_5, u_0)$. Fig. 3 ac-

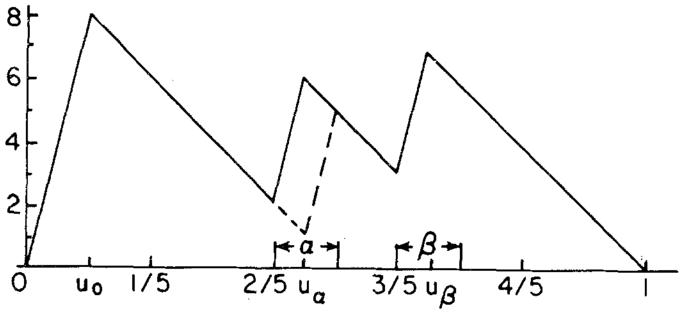

Fig. 3.

tually shows the construction of Theorem 1.10 for $u_0 = \frac{1}{10}$. It is easily verified directly that the two complementary intervals α and β are the only two on which $\pi(u) + \pi(u_0 - u) = 1$ does not hold and that $2\pi_\alpha(u_\alpha) \geq \pi(2u_\alpha)$. Here, $u_\alpha = \frac{9}{20}$.

Fig. 4 shows another example for $n = 5$ and $u_0 \in (\frac{1}{5}, \frac{2}{5})$. Its reflection is, again, another example. This figure is face 6 from [4, Table 2.2]. As in Fig. 3, α and β are the only two intervals of the second type, and $2\pi_\alpha(u_\alpha) \geq \pi(2u_\alpha)$.

In both Figs. 3 and 4, the role of α and β can be reversed, and we still have $2\pi_\alpha(u_\alpha) \geq \pi(2u_\alpha)$. In other words, if π_β is defined analogously to π_α with $u_\beta = \frac{13}{20}$ in Fig. 3 and $u_\beta = \frac{15}{20}$ in Fig. 4, then $2\pi_\beta(u_\beta) \geq \pi(2u_\beta)$. The next theorem shows that in this case a great many extreme valid inequalities can be generated which differ from π only in the intervals α and β.

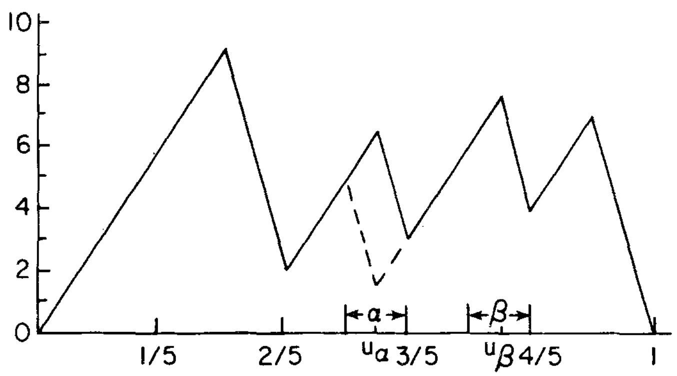

Fig. 4.

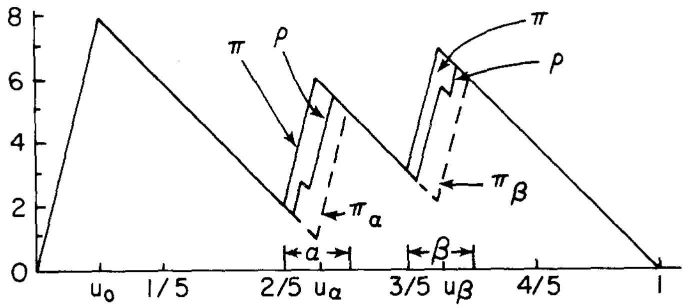

Fig. 5.

Theorem 4.5. *If α and β are the only two complementary intervals of the second type and if π_α and π_β are each valid inequalities for* $P(I, u_0)$, *then any continuous, piecewise linear function ρ on I having only the two slopes π^+ and π^- satisfying*

$$\rho(u) = \pi(u) , \qquad u \in I \backslash (\alpha \cup \beta) ,$$

$$\rho(u) = 1 - \rho(u_0 - u) , \quad u \in \alpha ,$$

is an extreme valid inequality for $P(I, u_0)$.

Fig. 5 illustrates such a function ρ in the example shown in Fig. 3.

Proof. We will consider only the case previously considered; that is, $\alpha = (g_i, u_{i+1})$ so that the left end-point of α is in G_n. Fig. 3 is this case, but Fig. 4 is not. The case $\alpha = (u_i, g_i)$ is similar and will not be considered.

First, we will show that neither α nor β is a subinterval of $(0, g_1)$ or $(g_{n-1}, 1)$. Since α and β have an element of G_n as left end-point, if either was a subinterval of $(0, g_1)$, then it would have to be $(0, u_1)$. However, this interval is of the first type as was remarked before Lemma 4.1. Hence the only possibility is that α or β is (g_{n-1}, u_n). We will now exclude that possibility.

Corollary 4.4 says that π_α is extreme and hence subadditive. We will show that π then is linear on $(g_{n-1}, 1)$ with a slope $-\pi^-$, and hence neither α nor β could be (g_{n-1}, u_n). To see that π is linear on $(g_{n-1}, 1)$, recall that

$$\pi_\alpha(g_i) + \pi_\alpha(u_0 - g_i) = \pi(g_i) + \pi(u_0 - g_i) = 1 = \pi(u_0)$$

and that π_α and π are decreasing on $(u_0 - g_i, u_0 - u_i)$ because of the shape of π in β. Hence,

$$\pi_\alpha(g_i) + \pi_\alpha(u_0 - u_i) = \pi_\alpha(R(u_0)) .$$

But

$$\pi_\alpha(u_\alpha) = \pi_\alpha(g_i) - \pi^-(|u_\alpha - g_i|) ,$$

and by subadditivity, one of the following inequalities holds:

$$\pi_\alpha(u_\alpha) + \pi_\alpha(u_0 - u_i) \geq \pi_\alpha(R(u_0) - (u_\alpha - g_i)) ,$$

$$\pi_\alpha(g_i) - \pi^-(|u - g_i|) + \pi_\alpha(u_0 - u_i) \geq \pi_\alpha(R(u_0) - (u_\alpha - g_i)) ,$$

$$\pi_\alpha(R(u_0)) - \pi^-(|u_\alpha - g_i|) \geq \pi_\alpha(R(u_0) - (u_\alpha - g_i)) .$$

By π_α having only two slopes, the reverse inequality also holds, and hence π_α is decreasing on the entire interval $(R(u_0), R(u_0) + g_1)$. This fact and subadditivity imply that π_α is decreasing on the entire interval $(g_{n-1}, 1)$, completing the proof that neither α nor β is a subinterval of $(g_{n-1}, 1)$ or $(0, g_1)$.

To return to the proof of the theorem, by Lemma 4.1 we can prove that ρ is a valid inequality by showing $\rho(u) + \rho(u_0 - u) \geq 1$ and $\rho(u) + \rho(v) \geq \rho(u + v)$ for $u, v \in \alpha \cup \beta$. The first inequality is obvious from the construction of ρ. What remains is to establish $\rho(u) + \rho(v) \geq \rho(u + v)$ for $u, v \in \alpha \cup \beta$. There are two cases: (i) u, v both in α (or both in β), and (ii) $u \in \alpha$ and $v \in \beta$.

In case (i), we only consider u, v both in α since both in β is exactly similar. By ρ being continuous with the same two slopes as π_α,

$$\rho(u) + \rho(v) \geq \pi_\alpha(u) + \pi_\alpha(v) .$$

By π_α being valid, and hence extreme,

$$\pi_\alpha(u) + \pi_\alpha(v) \geq \pi(u + v) .$$

If $u + v \in I \backslash (\alpha \cup \beta)$, then $\pi_\alpha(u + v) = \rho(u + v)$, so $\rho(u) + \rho(v) \geq \rho(u + v)$. If $u + v \in \beta$, then $\pi_\alpha(u + v) = \pi(u + v) \geq \rho(u + v)$, so again $\rho(u) + \rho(v) \geq \rho(u + v)$. The third subcase $u + v \in \alpha$ is excluded by α not being a subinterval of $(0, g_1)$ or $(g_{n-1}, 1)$. For any α which is a subinterval of (g_i, g_{i+1}), but not $(0, g_1)$ or $(g_{n-1}, 1)$, $u + v \notin \alpha$ when $u \in \alpha$ and $v \in \alpha$.

Next we consider case (ii), $u \in \alpha$ and $v \in \beta$. In this case there are two subcases: $|v| \geq |u_0 - u|$, and $|v| < |u_0 - u|$. First, consider $|v| \geq |u_0 - u|$. Since v and $u_0 - u$ are both in β,

$$|v - (u_0 - u)| \leq |R(u_0) - u_0|$$

$$\rho(u + v) = \rho(u_0 + (v - (u_0 - u))) = 1 - \pi^-(|v - (u_0 - u)|).$$

Hence we need only show

$$\rho(u) + \rho(v) \geq 1 - \pi^-(|v - (u_0 - u)|) .$$

But ρ has only two slopes, so

$$\rho(v) \geq \rho(u_0 - u) - \pi^-(|v - (u_0 - u)|) ,$$

$$\rho(u) + \rho(v) \geq \rho(u) + \rho(u_0 - u) - \pi^-(|v - (u_0 - u)|)$$

$$\geq 1 - \pi^-(|v - (u_0 - u)|) ,$$

completing the proof in this subcase. Next consider $|v| < |u_0 - u|$. In a similar way, we can now show that

$$\rho(u + v) = 1 - \pi^+(|(u_0 - u) - v|) ,$$

$$\rho(v) \geq \rho(u_0 - u) - \pi^+(|(u_0 - u) - v|) .$$

Hence, as before,

$$\rho(u) + \rho(v) \geq \rho(u) + \rho(u_0 - u) - \pi^+(|(u_0 - u) - v|)$$

$$= 1 - \pi^+(|(u_0 - u) - v|) = \rho(u + v) .$$

Hence ρ is a valid inequality for $P(I, u_0)$. To show that it is an extreme valid inequality, we need only remark that $\rho(u) + \rho(u_0 - u) = 1$ and apply Corollary 3.4. The theorem is thus proven.

The development here can be extended to the case where there are several intervals of the second type. However, its present form suffices to show an exponential rate of growth for some of the polyhedra $P(G_n, g)$ of [3]. We show this fact by means of the following example.

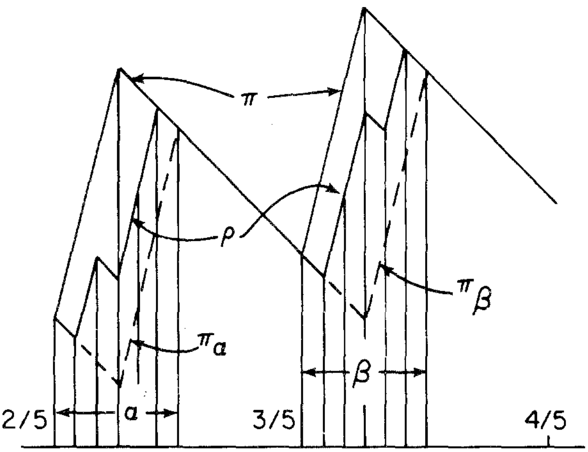

Fig. 6.

Example 4.6. Consider the group G_n for $n = 20K$, $K \geq 1$, and let $u_0 = \frac{1}{10} \in G_n$. We said that the function ρ in Fig. 5 gives an extreme valid inequality for $\mathrm{P}(I, u_0)$. The same is true for a great many functions ρ. In Fig. 6 we illustrate the intervals α and β from Fig. 5. Let us restrict ρ to be straight lines with breaks at points $k/20K$. In Fig. 6, $K = 3$, and we are perfectly free to let ρ have slope π^+ or $-\pi^-$ in the 3 intervals $(\frac{8}{20}, \frac{25}{60})$, $(\frac{25}{60}, \frac{26}{60})$, $(\frac{26}{60}, \frac{9}{20})$. The only restriction on ρ here is that it must have slope π^+ on as many intervals between $\frac{8}{20}$ and $\frac{10}{20}$ as on which it has slope π^-. Since ρ has been determined on $\frac{8}{20}$ to $\frac{10}{20}$, it is given on $\frac{12}{20}$ to $\frac{14}{20}$ by $\pi(u) + \pi(u_0 - u) = 1$. In general, there will be K intervals between $\frac{8}{20}$ and $\frac{9}{20}$ on which ρ can have either slope. Thus there are at least 2^K such functions ρ. By Theorem 3.2, each one is a face for the problem $\mathrm{P}(G_{20K}, \frac{1}{10})$. In fact, there are more than 2^K, namely $(2K)!/(K!K!)$, such functions ρ. This number results from the fact that we can choose any K of the $2K$ intervals between $\frac{8}{20}$ and $\frac{10}{20}$ for ρ to have slope π^+. As K becomes large, this number approaches $2^{2K}/\sqrt{(\pi K)}$ by Stirling's approximation of $n!$.

There is an abundance of such examples from [4, Table 2]. In particular, for $n = 7$, there are several similar cases. A similar construction works as long as there are two complimentary intervals α and β with π_α and π_β valid and provided the u_0 and all other break-points of π fall on group elements.

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