Now F_{m+1} is itself such an F. Hence, by induction, we have, for suitable g_k,

$$F_m = x + g_1 f + g_2 f^2 + \cdots + g_{m-1} f^{m-1}.$$

This suggests a change of variable to $y = f(x)$ (which is possible since $f'(X) \neq 0$). If $x = u(y)$ is the inverse function, and

$$\Phi_m(y) = F_m[u(y)], \qquad \psi_k(y) = g_k[u(y)],$$

then

$$\Phi_m = u + \psi_1 y + \psi_2 y^2 + \cdots + \psi_{m-1} y^{m-1}. \tag{B}$$

The conditions which must be satisfied by Φ_m are

$$\Phi(0) = X,$$

$$\Phi'(0) = \Phi''(0) = \cdots = \Phi^{m-1}(0) = 0.$$

The first of these is automatically satisfied.

Now a function $\Phi_m(y)$ satisfying these conditions is given immediately by the inverse Taylor expansion of $X = u(y - y)$. In fact, if we set $\Phi_m(y)$ equal to the sum of the first m terms of this expansion, viz.:

$$\Phi_m(y) = \sum_{k=0}^{m-1} \frac{(-y)^k}{k!} u^{(k)}(y),$$

then

$$X = \Phi_m(y) + \frac{(-y)^m}{m!} u^{(m)}(\eta),$$

and hence

$$\Phi_m(y) = X - \frac{(-y)^m}{m!} u^{(m)}(\eta)$$

satisfies the above conditions.

This method avoids the necessity of slapping down Bodewig's formula (14) or of motivating it by tedious experimenting with small values of m.

BOUNDARIES FOR THE LIMIT CYCLE OF VAN DER POL'S EQUATION*

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1. Introduction. In non-linear mechanics much interest centers on the Van der Pol (VDP) equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0 \tag{1}$$

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or its equivalent in the phase plane

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x + \mu(1 - x^2)y. \tag{2}$$

It is well known that (2) possesses a unique trajectory which represents a limit cycle in the sense of Poincaré. Using a different plane, La Salle[1] has located this limit cycle between two boundary curves in a very ingenious manner which however seems artificial and difficult to motivate. The present paper sets forth a simple and natural method for constructing outer and inner boundaries. The method admits of unlimited improvement but even its simplest application gives results superior to La Salle's in that the limit cycle is localized somewhat more sharply.

In the phase plane all trajectories other than the limit cycle spiral into it from the inside or the outside. The curves are described clockwise with increasing t. We use these facts to enclose the limit cycle between an outer boundary B_0 and an inner boundary B_i.

Introducing $r^2 = x^2 + y^2$, we transform (2) to

$$r\frac{dr}{dt} = \mu(1 - x^2)y^2, \qquad \frac{dx}{dt} = y. \tag{3}$$

Eliminating t, one obtains for the trajectories

$$r\frac{dr}{dx} = \mu(1 - x^2)y. \tag{4}$$

Since the field of (4) is symmetrical in the origin, it is sufficient to discuss solutions in the upper half-plane ($y \geqq 0$).

If C is a curve, $r = F(x)$, $y \geqq 0$, which intersects the x-axis only at $(-a, 0)$ and $(a, 0)$ and if at every point the value of dr/dx for C is *greater than* or equal to that of the VDP solution through that point, all VDP curves intersect C from above to below. Then C together with its image in the origin forms an *outer* boundary B_0.

The construction of an *inner* boundary B_i requires the substitution of *less than* for *greater than* in the above statement.

2. The outer boundary. To construct an outer boundary B_0, write (4) in the form

$$r\frac{dr}{dx} = \mu(1 - x^2)(r^2 - x^2)^{1/2}, \qquad y \geq 0, \tag{4'}$$

On an x-interval within which $1 - x^2$ is positive, replace x^2 under the radical by $(x^2)_{\min}$, its least value on the interval. The curves defined by the solutions of

$$r\frac{dr}{dx} = \mu(1 - x^2)[r^2 - (x^2)_{\min}]^{1/2}, \qquad y \geq 0, \tag{5}$$

have at every point of the interval a value of dr/dx greater than or equal to that of the corresponding VDP curve.

Similarly, on an x-interval within which $1 - x^2$ is negative, replace x^2 under the radical by $(x^2)_{\max}$, its largest value on the interval. The curves defined by

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[1] J. La Salle, *Relaxation oscillations*, Q. Appl. Math. **7**, 1-19 (1949). If La Salle's co-ordinates are (x, u), the relation to ours is given by $y/\mu + u = x - (x^3/3)$. His t is μ times ours.

$$r\frac{dr}{dx} = \mu(1 - x^2)[r^2 - (x^2)_{\max}]^{1/2}, \qquad y \geq 0, \tag{6}$$

have at every point of the interval a value of dr/dx greater than or equal to that of the corresponding VDP curve.

It remains to join together solutions of (5) and (6), valid over different intervals, to generate a continuous curve which will serve as C, the portion of B_0 in the upper half-plane.

This boundary, at least for large μ, may be expected to lie rather close to the limit cycle since over much of the short x range, x^2 is small relative to r^2 so that the error made by the proposed substitution is not serious. In fact, by using a large number of intervals a very accurate outer boundary may be constructed. But even the simplest outer boundary, using three intervals, is surprisingly good. We proceed to the details for this case.

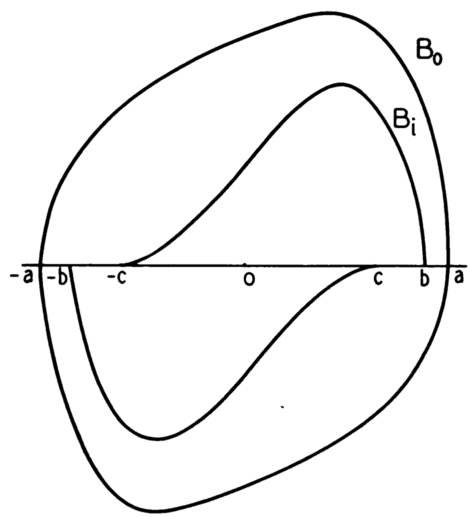

FIG. 1. Sketch of Boundaries.

Let C intersect the x-axis at $-a$ and a. The intervals to be used are $[-a, -1]$, $[-1, 1]$ and $[1, a]$. We start at $-a$ and work across to a in the upper half plane. The equations become

$$[-a, -1] \qquad r\frac{dr}{dx} = \mu(1 - x^2)(r^2 - a^2)^{1/2},$$

$$[-1, 1] \qquad r\frac{dr}{dx} = \mu(1 - x^2)(r^2 - 0^2)^{1/2},$$

$$[1, a] \qquad r\frac{dr}{dx} = \mu(1 - x^2)(r^2 - a^2)^{1/2}.$$

Each of these equations is separable and possesses a general and a singular solution. We use the singular solution $r = a$ for $[-a, -1]$ since it is the only solution through

$(-a, 0)$ which is real in the interval. For the other intervals, we use the general solutions which take the form

$$(r^2 - c^2)^{1/2} = \mu\left(x - \frac{x^3}{3}\right) + k.$$

It is easy to choose the constants k to secure continuity.

The principal interest is in the amplitude a which determines the size of the enclosure. To find a, we take definite integrals successively over the intervals $[-a, -1]$, $[-1, 1]$ and $[1, a]$, finding in each case the relation between r at the left side and r at the right side of the interval. The results are

$$r(-1) - a = 0,$$

$$r(1) - r(-1) = \frac{4\mu}{3}, \tag{7}$$

$$[r^2(1) - a^2]^{1/2} = \frac{\mu}{3}(a^3 - 3a + 2).$$

Combining

$$a^3 - 3a + 2 - 4\left(1 + \frac{3a}{2\mu}\right)^{1/2} = 0.$$

Thus even in the simplest case there is a slight improvement over La Salle's result, which in this form is

$$a^3 - 3a + 2 - 4\left(1 + \frac{3a}{4\mu}\right) = 0.$$

We quote some numerical results for $\mu = 3$. La Salle's boundary gives $a = 2.21^+$. Our result is $a = 2.18^-$. If seven intervals are used (joining at $-3^{1/2}$, -1.6, -1.4, -1, 1 and $3^{1/2}$), one obtains $a = 2.10^-$.

3. The inner boundary. To construct an *inner* boundary B_i, use

$$r\frac{dr}{dx} = \mu(1 - x^2)(r^2 - x^2)^{1/2}, \qquad y \geq 0, \tag{4'}$$

as before, but replace x^2 under the radical by $(x^2)_{max}$ if $1 - x^2$ is positive within the x-interval, by $(x^2)_{min}$ if $1 - x^2$ is negative there.

Using three intervals $[-c, -1]$, $[-1, 1]$ and $[1, b]$, we obtain the equation

$$r\frac{dr}{dx} = \mu(1 - x^2)(r^2 - 1)^{1/2},$$

applicable to all three intervals. One is tempted to set $c = b$ and reflect in the origin· There is however a difficulty. To the left of $x = -1$, dr/dx is negative but $-dr/dx \leqq 1$ for $x \geqq -c$. This condition restricts the choice of c.

Since $r(-dr/dx) = \mu(x^2 - 1)(r^2 - 1)^{1/2}$, $y \geqq 0$, and since on the x-axis, $r = c$, it follows that

$$c \geq \mu(c^2 - 1)^{3/2}.$$

The largest c corresponds to the equality. If

$$d = \mu(d^2 - 1)^{3/2}$$

defines d for a given μ, we therefore choose $c \leqq d$ and follow the curve along the upper half-plane to its right-hand intersection at b. Reflection of this curve in the origin leaves two gaps, one between $-b$ and $-c$ and one between c and b. If $b > c$ we may use segments of the x-axis to complete the inner boundary. Since the VDP curves are described clockwise with increasing t, these curves cross the added segments in the required direction.

The relation between $-c$ and b is given by integrating between these limits and is

$$(b^2 - 1)^{1/2} - (c^2 - 1)^{1/2} = \mu\left(b - \frac{b^3}{3} + c - \frac{c^3}{3}\right). \tag{9}$$

We know that $c \leqq d$. The largest inner boundary using three intervals arises from the choice $c = d$. However, for the sake of simplicity, we may obtain an inner boundary valid for all μ by choosing $c = 1$. Then (9) becomes

$$b^3 - 3b - 2 + \frac{3}{\mu}(b^2 - 1)^{1/2} = 0,$$

which is a slight improvement over La Salle's result

$$b^3 - 3b - 2 + \frac{3b}{\mu} = 0.$$

For $\mu = 3$, for example, La Salle's result is $b = 1.77$. Ours is $b = 1.81$ with $c = 1$ and $b = 1.87^+$ with $c = d = 1.248$. If additional points of division are placed at $-.5$, $.5$ and $3^{1/2}$, $b = 1.94$ is obtained.

4. Conclusion. In conclusion, it is clear that the simplicity of the calculations makes it relatively easy to obtain indefinitely better boundaries by increasing the number of intervals. Over these intervals, the form of the solution is (with one exception) always the same, different $(x)^2_{\max}$ and $(x)^2_{\min}$ being inserted. It is therefore sufficient to be armed with a table of square roots and cubes to find a or b by trial solution. With a moderate amount of work, it is fortunately possible to supplement the method of perturbations which is useful for $\mu \ll 1$ and the known results for $\mu \to \infty$ by giving a good account of the limit cycle in the intermediate range of μ.

It is also obvious that the method of this paper can be applied if $x^2 - 1$ in Eq. (1) is replaced by other suitable functions $f(x)$. Further generalizations are possible but will not be discussed.